SHELL MODEL OF MAGNETIC FIELD EVOLUTION UNDER THE HALL EFFECT

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The Hall effect occurs in strongly magnetized conductive media and results in nondissipative currents directed normally to the electric field. We consider the magnetic field dynamics driven by the Hall effect ignoring fluid motions. Electronic magnetohydrodynamic turbulence may result from a nonlinear evolution of the magnetic field. We introduce a shell model for the corresponding induction equation written for the poloidal and toroidal components. The model provides the conservation laws in dissipationless limit, where the energy and the magnetic helicity become integrals of motion. Numerical simulations confirm that the inertial range with the spectral law \( -7/3 \) is established in the EMHD system provided the Hall parameter \( R_H \) is sufficiently large and the outflow of helicity at the left end of the inertial range is possible.

1. Introduction. Electric conductivity of strongly magnetized media is anisotropic, depends on the magnetic field strength and introduces nonlinearities in the induction equation. The nonlinearities are known as the Hall-drift and the ambipolar diffusion. The focus of this work is the investigation of the influence of the first one on the magnetic field evolution ignoring fluid motions. The induction equation of strongly magnetized media with the Hall effect [1] can be written in the dimensionless form as

\[
\frac{\partial B}{\partial t} = R_H^{-1} \Delta B - \nabla \times \left[ (\nabla \times B) \times B \right],
\]

where \( R_H = B_0/B_e \) is the Hall parameter, \( B_0 \) is the characteristic magnetic field strength, and \( B_e \) corresponds to the level of the magnetic field strength when the Hall-drift becomes essential. The latter is estimated as \( B_e = m_e^* c / \tau_e \), where \( e \) is the elementary charge, \( m_e^* \) is the effective mass of an electron and \( \tau_e \) is the electron relaxation time.

The effect of the Hall-drift on the magnetic field evolution has been considered by a number of authors. Most of the discussions deal with the influence of the Hall effect on the magnetic field evolution in the core of neutron stars. It was shown that in the case of a nondissipative Hall-drift the field decay rate increases due to the Hall effect [2]. However, the redistribution of an initial dipolar magnetic field into small scales is still an open question.

The next point to be discussed concerns the possibility for inertial energy transfer along the spectrum provided the Hall parameter \( R_H \) is sufficiently large. The idea of Hall cascade was mentioned in [3]. The electronic magnetohydrodynamics (EMHD) refers to a branch of plasma oscillations defined by predominance of the Hall term, and is a limiting case of multicomponent MHD, where the motion of ions can be neglected and the motion of electrons preserves quasineutrality. The properties of the EMHD turbulence were discussed in [4, 5].

In the limit of high \( R_H \), equation (1) gets two integrals of motion, which must be conserved by the system. These are the total energy \( E \) and the magnetic
helicity $H$

\begin{align*}
E &= \int \mathbf{B}^2 dV, \\
H &= \int \mathbf{B} \cdot \mathbf{A} dV,
\end{align*}

(2)

where $\mathbf{A}$ is the vector potential.

The theory for high Reynolds number turbulence originated from the famous Kolmogorov "5/3" law. Kolmogorov’s scenario implies a large range of scales between the macroscale $l_f$, characterizing the input of energy (this scale contains the main part of energy), and the microscale $l_\nu$, characterizing the scale of viscous dissipation. In this range, called the inertial range, the energy transfer is due to the local nonlinear interactions. The notion "local" means that the process is determined by interactions of vortices of similar scales, and the energy transfer proceeds scale by scale (this transport is called the cascade). The spectral flux of energy at any scale inside the inertial range must be constant and equal to the energy dissipation rate $\varepsilon$. Similar arguments for the magnetic energy transfer, described by Eq.(1), read

\begin{align*}
\delta E_i \approx \frac{\delta B_i^2}{l^2} = \frac{\delta B_i^3}{l^2} = \varepsilon_B.
\end{align*}

Then

\begin{align*}
\delta B_i \approx (\varepsilon_B l^2)^{1/3} = \varepsilon_B^{1/3} l^{2/3},
\end{align*}

(4)

and the spectrum obeys the "7/3" law

\begin{align*}
E_B(k) = C_E \varepsilon_B^{2/3} k^{-7/3}.
\end{align*}

(5)

Spectral law (5) with a direct spectral flux (toward the small scales) is expected if the second integral vanishes ($H \approx 0$). If the flux of magnetic helicity dominates in the cascade, similar dimensional arguments lead to the spectral law "5/3" [4]

\begin{align*}
E_B(k) = C_H \eta_B^{2/3} k^{-5/3},
\end{align*}

(6)

where $\eta_B$ is the averaged flux of magnetic helicity. Let us emphasize that in contrast to usual hydrodynamical turbulence the "5/3" law corresponds to the cascade of magnetic helicity. If both conserved quantities (2) are essential in the inertial range of scales (scales $l_i$ in which the variations of the magnetic field remain sufficiently large $\delta B_i \gg \text{Rh}^{-1}$), the nonlinear interactions can cause their transfer toward the opposite ends of the spectrum. General dimensional arguments suggest that the energy should follow the direct cascade whereas the helicity is transferred to larger scales.

The first attempt to study the Hall cascade numerically using the shell model of equation (1) was undertaken in [6]. The model was written for variables $B_n$, each of them describing all magnetic field oscillations in the corresponding shell of wave numbers. It was a shell model, in which the helicity is described by a quadratic quantity, the sign of which is determined by the shell number (even shells provide a positive helicity and the odds provide a negative one). It was shown that the direct energy cascade with a "7/3" spectral law was really established in the system producing a burst of energy transport to small scales, yet during a short time interval only, after which it became blocked.
The axisymmetric magnetic field can be split into poloidal and toroidal components, for which the induction equations in the dimensionless form are

\[ \frac{\partial B^p}{\partial t} = R_{\text{H}}^{-1} \Delta B^p - \nabla \times \left[ (\nabla \times B^t) \times B^p \right], \quad (7) \]

\[ \frac{\partial B^t}{\partial t} = R_{\text{H}}^{-1} \Delta B^t - \nabla \times \left[ (\nabla \times B^p) \times B^p + (\nabla \times B^t) \times B^t \right]. \quad (8) \]

As the Hall effect is nonlinear, different components are coupled. For instance, a toroidal field will be generated as a result of the Hall current associated with term \((\nabla \times B^p) \times B^p\) even if the initial magnetic configuration is pure poloidal. In its turn, a toroidal field contributes to the generation of a poloidal one. It is still unclear whether the Hall effect can generate a poloidal field if the original magnetic configuration is purely toroidal. For this case the instability of a small-scale field was found in the presence of the background field using the linear analysis only [7]. The available computational facilities do not allow one to study the essentially nonlinear regime by direct numerical methods.

Below, we construct a shell model with the Hall effect for poloidal and toroidal components of a magnetic field. In this model the helicity can be described through the production of both field components of a given scale avoiding the artificial definition of helicity used in traditional shell models, which separates the positive and negative helicities in different shells. We apply our model to study the nonlinear interaction of the components and to analyze the cascade process in an essentially nonlinear regime.

**2. Shell model.** Shell models were introduced in the 70-ies as an attempt to describe the energy cascade in fully developed turbulence by the dynamical system of relatively small number of variables. In spite of their simplicity and apparent inadequacy for complicated space-temporal structure of real turbulent flows, these models describe some very specific properties of high Reynolds number turbulence and have become very popular in the last decade [8]. Inter alia, this kind of models has been developed for MHD-turbulence [9].

The shell model is intended for describing the cascade process in a large range of scales (wave numbers) by a chain of variables. Each variable characterizes all velocity oscillations with wave numbers \(k\) ranging between \(k_n = k_0 \lambda^n\) and \(k_{n+1}\) (it is a shell of wave numbers). The parameter \(\lambda\) characterizes the ratio between two adjacent scales (shell width). The model includes a set of ordinary differential equations, which should reproduce the basic properties of the motion equations. The model has to describe a square nonlinearity and to conserve the integrals of motion in the dissipationless limit.

A simple shell model for Eqs.(1) can be written in the form [6]

\[ d_t B_n + R_{\text{H}}^{-1} k_n^2 B_n = k_n^2 \left( B_{n+2} B_{n+1} + \frac{\lambda}{\lambda^2} B_{n+1} B_{n-1} - \frac{1}{\lambda^3} B_{n-1} B_{n-2} \right), \quad (9) \]

In the limit \(R_{\text{H}} \to \infty\) this set of equations conserves two quantities

\[ W_1 = \sum_n k_n^2 |B_n|^2, \quad W_2 = \sum_n (-1)^n k_n^{-1} |B_n|^2, \]

which are considered as energy and helicity.
Now we introduce a shell model corresponding to Eqs. (7-8)

\[
\begin{align*}
\dot{B}_n^p + R_H^{-1} k_n^2 B_n^p &= k_n^2 \left( B_{n+2}^p B_{n+1}^t - \frac{(\lambda + 1)}{\lambda^2} B_{n+1}^p B_{n-1}^t + \frac{1}{\lambda^3} B_{n-1}^p B_{n-2}^t \right) + \\
\dot{B}_n^t + R_H^{-1} k_n^2 B_n^t &= k_n^2 \left( B_{n+2}^t B_{n+1}^p - \frac{(\lambda + 1)}{\lambda^2} B_{n+1}^t B_{n-1}^p + \frac{1}{\lambda^3} B_{n-1}^t B_{n-2}^p \right) + \\
&\text{for } n_{\text{min}} \leq n \leq n_{\text{max}}.
\end{align*}
\]

These equations describe the evolution of poloidal and toroidal component oscillations $B_n^p$ and $B_n^t$ in the corresponding shell of wave numbers, defining the magnetic energy and helicity as sums

\[
E = \sum_{n=n_{\text{min}}}^{n_{\text{max}}} B_n^p B_n^p, \quad H = \sum_{n=n_{\text{min}}}^{n_{\text{max}}} k_n^{-1} B_n^p B_n^t.
\]

Conservation is provided for any value of the parameter $\lambda$. A remarkable advantage of the present shell model is a more natural definition of helicity. Now we have no need to introduce the artificial factor $(-1)^n$ to obtain a quadratic quantity with an arbitrary sign. The second advantage is the possibility of describing the energy exchange between the poloidal and toroidal components of a magnetic field.

We focus on the most important phenomena of the spectral cascade caused by the Hall effect, which is more pronounced at large $R_H$. However, we are keeping in mind the applications to astrophysical objects. For neutron stars and white dwarfs, the Hall parameter is estimated as $R_H \sim 10^2$. So we consider $10 \cdot 10^2 \leq R_H \leq 10^3$. In our simulations, $\lambda = (1 + \sqrt{5})/2$ (golden section) and the range of shells $n_{\text{max}} - n_{\text{min}} \approx 40$.

The integration has been done by the fourth-order Runge–Kutta method with an adaptive time step. We apply this routine to the system of ordinary differential Eqs. (10) after substitution $B_n(t) = H_n(t) \exp(-k_n^2 R_H^{-1} t)$. It allows to avoid the numerical effect of negative viscosity and to increase the time step.

3. Free decay. We study the problem of how the Hall effect affects the energy evolution of poloidal and toroidal magnetic field components in a free de-
caying system. The initial state \((t = 0)\) corresponds to the energy concentration in two largest scales \(k_0 = 1\) and \(k_1 = \lambda\). The coupling of poloidal and toroidal modes is more pronounced in the case of similar values of initial energies. The magnetic energy evolution for \(R_H = 10^2\), obtained from numerical simulations of (10), is shown in Fig. 1a. The thick solid line corresponds to the total energy and thick dashed line corresponds to helicity. Thin solid and dashed lines show the energy evolution of the poloidal and the toroidal components. One can see that after few oscillations, the two components evolve almost independently. Fig. 1b illustrates the energy dissipation rate. It is evident that the Hall effect essentially accelerates the energy dissipation at the early stage of evolution.

Three stages are readily seen in the energy evolution. The first stage corresponds to the formation of the energy cascade, which is established already at \(t \approx 0.3\). The energy spectrum is characterized by a very steep slope (see Fig. 2). This moment corresponds to the maximal value of the energy dissipation rate (see Fig. 2) and to a week energy exchange between the poloidal and toroidal components. The second stage is characterized by a slow decrease of the spectral energy flux and a corresponding decrease of the energy dissipation rate. This stage continues up to \(t \approx 8.0\) (see Fig. 1b), when the energy cascade really ceases. After that all modes decay mainly due to the ohmic dissipation (this is the last stage of evolution).

The spectral density of helicity at large wave numbers can be higher than the spectral density energy \((H(k) \leq k E(k))\). So, the energy cascade at small scales may be controlled by the distribution of the magnetic helicity. To check this, we consider the initial configuration, which corresponds to \(H = 0\). In this case we found that the magnetic energy dissipated in each shell, and the spectral energy
flux vanished (see Fig. 3). Note that at the beginning of evolution the helicity grows while the energy decays.

We analyzed the evolution of a weak poloidal field at a strong initial toroidal one. In case $R_H = 10^2$ the poloidal energy does not grow (see Fig. 4a). The poloidal magnetic field generation is observed at a larger value of the Hall parameter ($R_H = 10^3$, see Fig. 4b). In the latter case the "helicoidal oscillations" of the poloidal and toroidal components are also more pronounced.

Now we study the inertial range formation at a large value of the Hall parameter. In Fig. 5 the results of three numerical experiments for $R_H = 10^4$ are shown. In all cases the initial conditions are the same and $H \approx E$. In the first experiment, the range of scales $0 \leq n \leq 30$ was considered. Here, there is no real inertial range in the spectrum (Fig. 5a). The outflow of energy from the largest scale is blocked by the requirement of helicity conservation. The situation is different if helicity can move to larger scales. Fig. 5b shows the results of simulations under the same initial conditions, but in an extended range of scales ($-4 \leq n \leq 30$). In this case an inertial range with a spectral slope $-7/3$ is visible up to $k \approx 100$ and the energy of the shell number $n = 0$ (where the initial energy was concentrated) decays like in the neighboring shells. Note that the dissipation scale can be evaluated as $k_{\text{dis}} \approx 10^4$ (just at this point the spectral density falls down). So a large range of scales exists between the inertial range and the dissipative one. This "intermediate dissipative range" arises due to the fact that the dissipation term and the nonlinear term in Eqs. (10) have the same prefactor $k_n^2$ (in the Navier–Stokes equation the nonlinear term has the prefactor $k$). To obtain a well pronounced inertial range we have applied the idea of "hiperviscosity", which was widely used in numerical simulations of two-dimensional turbulence to extend the inertial range of enstrophy transport [10]. Following this idea, the dissipation term $\sim k_n^2 B_n$ is replaced by a term like $\sim (k_n/k_{\text{dis}})^m B_n$ with a typical value $m = 4 \div 8$. The results obtained for $m = 4$ are shown in Fig. 5c, where the inertial range with the spectral law $-7/3$ really extends up to $k \approx 10^5$.

Fig. 5d, which corresponds to the first numerical experiment (Fig. 5a), illustrates two facts. First, at a high Hall parameter the helicoidal oscillations become more intensive (compare with Fig. 1a). Second, in this case the dissipation of helicity practically vanishes (the corresponding line is horizontal).

4. Conclusions. The shell model of EMHD turbulence based on the separate scale-wise description of the poloidal and toroidal magnetic fields has been introduced. This model allows one to describe the helicity in a more natural way than in the known shell models and to study the evolution and interaction of
two components of the magnetic field. The model correctly reproduces the basic features of the magnetic field evolution under the Hall effect.

Numerical simulations confirm that the inertial range with the spectral law \( -7/3 \) is established in the EMHD system provided the Hall parameter \( R_H \) is sufficiently large and the outflow of helicity at the left end of the inertial range is possible. We have shown that the poloidal field develops in an initially pure toroidal configuration.

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