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The Cross-Helicity Effect on Cascade Processes in MHD Turbulence

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The cross helicity $H = \langle \mathbf{v} \cdot \mathbf{b} \rangle$ characterizes the level of correlation between pulsations of the magnetic field $\mathbf{b}$ and the velocity field $\mathbf{v}$. In the ideal three-dimensional magnetic hydrodynamics, it is the integral of motion alongside with the total energy $E^T = E^v + E^b$, $E^v = \langle \frac{|\mathbf{v}|^2}{2} \rangle$, and $E^b = \langle \frac{|\mathbf{b}|^2}{2} \rangle$. The third integral of motion is the magnetic helicity; but within the framework of this study, we consider the fields in which the average magnetic helicity is close to zero and, thus, it is not considered here.

The representation about the developed turbulence as a random process makes it possible to expect that, if there are no special reasons, the developed conducting-fluid turbulence (the MHD turbulence) should be characterized by a low cross-helicity level. Exactly such a situation is usually considered. Interest in cross helicity arose after highly correlated pulsations of velocity and magnetic field were found in the solar wind [1]. Analysis of the energy and helicity evolution in a freely degenerating MHD turbulence showed that the helicity decays more slowly than the energy; hence, the degree of correlation of fields $\mathbf{v}$ and $\mathbf{b}$ determined by the correlation coefficient $C = H/E^T$ can increase in time for the free degeneration [2].

By itself, MHD turbulence gives the possibility of developing various scenarios. The specificity of the conducting-fluid hydrodynamics is the possibility of occurrence of Alfvén waves; it is assumed they play a key role in the turbulent cascade, which leads to the Iroshnikov–Kraichnan spectral law $E^b(k) \sim E^v(k) \sim k^{-3/2}$ [3, 4]. In the calculations on the mesh (512^3) [5], it was shown that no Kraichnan–Iroshnikov spectrum arises in the noncorrelated turbulence without an external field, and the turbulence with the spectrum close to $E^b(k) \sim E^v(k) \sim k^{-5/3}$ is realized in the inertial interval. In [6], the mesh was expanded to 1024^3, and the turbulence was considered both with and without the imposed external field. For the MHD turbulence without the external field, the "–5/3" law was confirmed and a significant anisotropy for which the transverse pulsations follow the Iroshnikov–Kraichnan law was revealed in the external field.

The problem of the cross-helicity effect on the forced MHD turbulence was considered in [7] only in the context of the Alfvén scenario (i.e., the turbulence that gives the –3/2 spectrum without the cross-helicity source). On the basis of the quasi-normal closing, it was shown that the system tends to the steady state in which the correlation coefficient proves to be much higher than the ratio between the power of the helicity and energy sources. In this case, the energy spectrum becomes more abrupt.

The purpose of this work is to investigate the spectral properties of the developed isotropic (non-Alfvén) MHD turbulence continually excited by an external force, which also introduces the cross helicity into the flow alongside with the energy.

We consider MHD turbulence with the magnetic Prandtl number equal to about unity. On the integral scale L, external forces with a specific power equal to ε operate. The same forces introduce a certain specific cross helicity χ in the turbulent flow and introduce no magnetic helicity. We assume that the equidistribution of kinetic and magnetic energy $\delta v_l^2 \approx \delta b_l^2 \approx E_l$ is fulfilled in the inertial interval. Within the limits of the inertial interval, the energy flux is constant on an arbitrary scale l and equal to the energy-dissipation rate (the specific power of external forces)

$$\frac{E_l}{t_l} = \varepsilon, \qquad (1)$$

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where t_l is the characteristic exchange time, which is taken to be equal to the vortex-revolution time $\tau_l \approx \frac{l}{\delta v_l}$ in the homogeneous-turbulence theory. In the case of noncorrelated velocity and magnetic-field pulsations, $H = 0$ and this estimate can be accepted also for the MHD turbulence in which nonlinear interactions dominate instead of the Alfvén waves.

The basic idea of further arguments is that the cross helicity introduced in the flow delays the exchange (increases the time)

$$t_l = \frac{l}{\delta v_l}\xi_l = \tau_l\xi_l \tag{2}$$

and the delay coefficient ξ_l is related to the correlation level for the velocity and magnetic-field pulsations on this scale. Hypothesis (2) leads to the estimate of energy pulsations on the scale l in the form

$$\delta v_l^2 \approx (\varepsilon l \xi_l)^{2/3}. \tag{3}$$

In this case, the delay coefficient actually determines the deviation from the Kolmogorov 4/5 law

$$\xi_l = \frac{\delta v_l^3}{\varepsilon l}, \tag{4}$$

and Eq. (3) coincides with the Kolmogorov–Obukhov law $\delta v_l^2 \approx (\varepsilon l)^{2/3}$ for $\xi_l = 1$.

The delay of the transport processes should lead to an increase in the turbulent-motion energy (in comparison with the noncorrelated turbulence for the same power source). The application of estimate (3) to the energy-transfer scale $l = L$ for which $v_L^2 \approx E$ gives

$$E \approx (\varepsilon L \xi_L)^{2/3}. \tag{5}$$

The simplest assumption about the form of ξ_L consists in the fact that, on the scales of action of external forces, the delay of the cascade transfer is determined by the quantity $(1 - \chi/\varepsilon)$, which is the measure of noncorrelation of the perturbations introduced by external forces. Taking into account the quadraticity of the terms describing the processes of spectral transfer, we can assume that $\xi_L \approx \left(1 - \frac{\chi}{\varepsilon}\right)^{-2}$, which gives the estimate of the average energy of the permanently excited turbulence

$$E \approx \frac{(\varepsilon L)^{2/3}}{\left(1 - \frac{\chi}{\varepsilon}\right)^{4/3}}. \tag{6}$$

Thus, the cross helicity hampers cascade energy transfer and leads to energy accumulation in the system. This accumulation proceeds until the vortex intensification compensates the decreasing efficiency of nonlinear interactions.

When assuming that the external forces introduce the cross helicity in the turbulence with the specific flux χ, it is necessary also to accept the hypothesis that the cross-helicity flux is constant over the spectrum for a steady state, which gives

$$\frac{H_l}{t_l^{(\chi)}} = \chi; \tag{7}$$

at that, the cross-helicity-exchange time $t_l^{(\chi)} = \tau_l \xi_l^{(\chi)}$ could not coincide with the energy-exchange time t_l.

Let C_l be the correlation coefficient for the velocity and magnetic-field pulsations on the scale l:

$$C_l = \frac{\langle \delta v_l \delta b_l \rangle}{\sqrt{\langle \delta v_l^2 \rangle \langle \delta b_l^2 \rangle}} \approx \frac{H_l}{E_l}, \tag{8}$$

where the angular brackets mean averaging and H_l is the cross helicity on this scale. Thus, $H_l \approx C_l E_l$ and the substitution in Eq. (7) gives

$$C_l \delta v_l^3 \approx \chi l \xi_l^{(\chi)}. \tag{9}$$

Comparing this expression with Eq. (3), we can relate the correlation coefficient to the characteristics of the exchange rate on the corresponding scale

$$C_l \approx \frac{\chi}{\varepsilon} \frac{\xi_l^{(\chi)}}{\xi_l}. \tag{10}$$

If the delay in the helicity and energy exchanges depends identically on the scale; i.e., $\xi_l^{(\chi)} \sim \xi_l$, the velocity and magnetic-field correlation should be independent of the scale (and, on the contrary, the correlation independence of the scale means the identical dependence of coefficients $\xi_l^{(\chi)}$ and ξ_l on the scale).

In the strongly correlated turbulence, the relation between the exchange times and the vortex-revolution time, which should be very large on the energy-transfer scale, decreases with the scale approaching unity on the dissipation scale. If it depends on the scale in the inertial interval by the power law

$$\xi_l \approx \xi_L \left(\frac{l}{L}\right)^{\mu}, \tag{11}$$

the correction of the spectral energy distribution of pulsations is unambiguously related to the parameter μ:

$$\delta v_l^2 \approx \varepsilon^{2/3} l^{2/3(1+\mu)}. \tag{12}$$

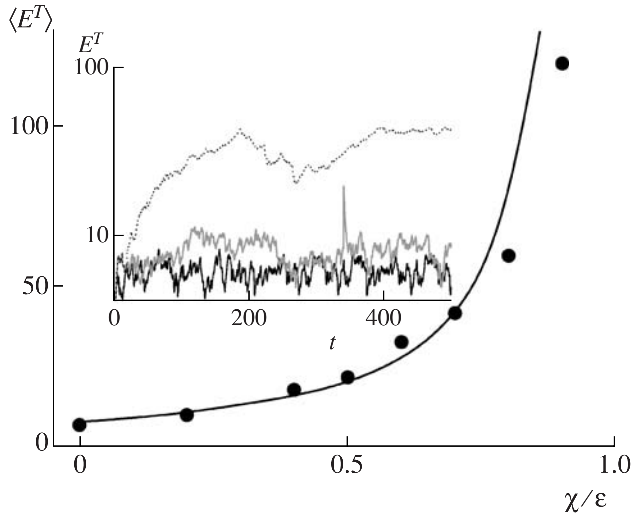

Fig. 1. Dependence of the average energy of the steadily excited MHD turbulence on the introduced cross helicity χ/ε. In the inset, the energy evolution is shown at $\chi = 0$ (the black line), $\chi = 0.3$ (the gray line), and $\chi = 0.6$ (the dotted line).

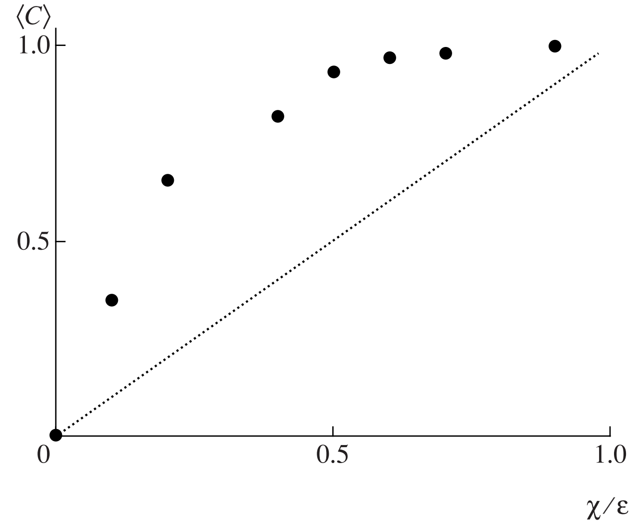

Fig. 2. Dependence of the average correlation level $C = \langle H\rangle / \langle E\rangle$ on the introduced cross-helicity level χ/ε.

The conclusions about the cross-helicity effect on the properties of the stationary MHD turbulence confirm the results of numerical calculations executed with use of the cascade model of MHD turbulence. The cascade models describe the processes of spectral transfer in the developed turbulence with the help of a small number of variables, each of which is the collective characteristic of amplitudes of pulsations of the velocity field U_n and the magnetic field B_n in the wavenumber interval $k_n < |\mathbf{k}| < k_{n+1}$, where $k_n = \lambda^n$ and λ is the interval (envelope) width. The equations for collective variables are written so that they could reproduce the "basic" properties of the initial equations of motion; that is, they have the same integrals of motion and the same type of nonlinearity. The cascade models are an efficient tool for investigating the statistical properties of the developed small-scale turbulence (see, for example, [8]); in particular, they reproduce well the basic known properties of MHD turbulence and the small-scale dynamo [9]. However, model [9] inherited the basic disadvantage of cascade models associated with the method of describing the helicity (in these models, the different-sign helicity is attributed to envelopes with even or odd numbers n). In this work, we used a new model, which is obtained by a generalization of the model proposed in [10] for the spiral turbulence on the MHD case. The equations of the model have the form

$$d_t U_n = ik_n(\Lambda_n(U, U) - \Lambda_n(B, B)) - \frac{k_n^2 U_n}{\mathrm{Re}} + f_n, \quad (13)$$

$$d_t B_n = ik_n(\Lambda_n(U, B) - \Lambda_n(B, U)) - \frac{k_n^2 B_n}{\mathrm{Rm}}, \quad (14)$$

where

$$\Lambda_n(X, Y) = \lambda^2(X_{n+1}Y_{n+1} + X^*_{n+1}Y^*_{n+1}) - X^r_{n-1}Y_n - X_nY^r_{n-1} + i\lambda(2X^*_nY^i_{n-1} + X^r_{n+1}Y^i_{n+1} - X^i_{n+1}Y^r_{n+1}) + X_{n-1}Y_{n-1} + X^*_{n-1}Y^*_{n-1} - \lambda^2(X^r_{n+1}Y_n + X_nY^r_{n+1}) + i\lambda(2X^*_nY^i_{n+1} + X^r_{n-1}Y^i_{n-1} - X^i_{n-1}Y^r_{n-1});$$

the asterisk designates conjugation, while the superscripts r and i are the real and imaginary parts. Without dissipation, the total energy $E^T = \Sigma(|U_n|^2 + |B_n|^2)/2$, the cross helicity $H = \Sigma(U_nB^*_n + B_nU^*_n)/2$, and the magnetic helicity $H_m = \Sigma k_n^{-1}((B^*_n)^2 - B_n^2)/2$. If the magnetic field is zero, the hydrodynamic helicity $H_h = \Sigma k_n((U^*_n)^2 - U_n^2)/2$ is conserved. The distinctive feature of the model is the possibility of the occurrence of arbitrary-sign helicity in an arbitrary wavenumber interval.

In all calculations, the Reynolds number and the magnetic Reynolds number $\mathrm{Re} = \mathrm{Rm} = 10^6$, and the width of spectral envelopes $\lambda = 1.618$. Time is measured in dimensionless units equal to the vortex-revolution time on the maximal scale. The force f_n operates only in the two highest layers (the greatest scales) providing constant pumping of kinetic energy $\varepsilon = 1$ and set pumping of cross helicity χ.

In Fig. 1, we show the time-average values of the total energy of the system obtained for various values of χ/ε and agreeing well with estimate (6) to which the solid line in the figure corresponds. In the same figure, we show how the total energy E^T of the system varies

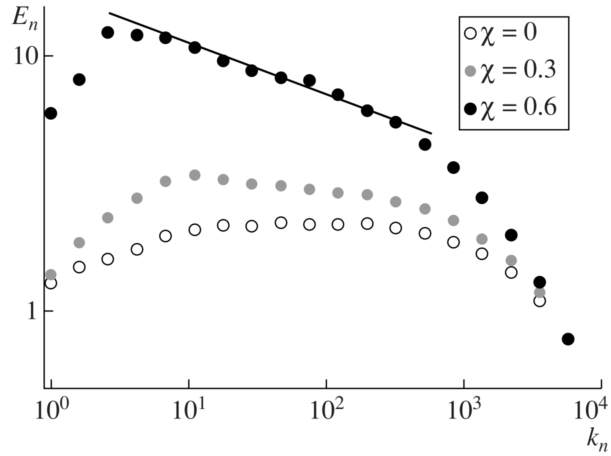

Fig. 3. Compensated energy spectra.

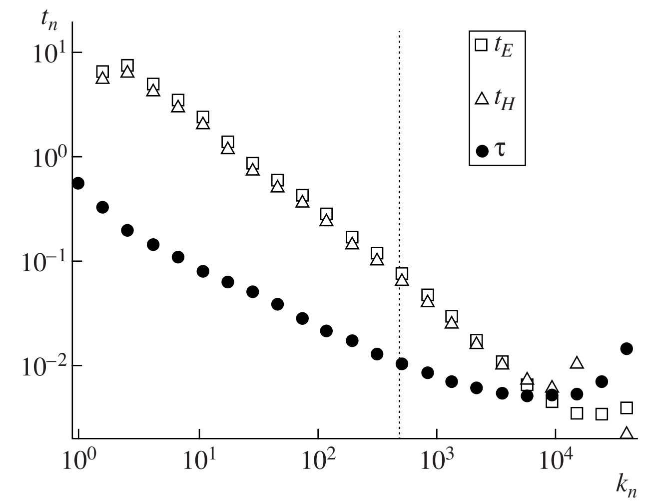

Fig. 4. Vortex-revolution and exchange times for the case of $\chi/\varepsilon = 0.6$. The vertical line corresponds to the inertial-interval boundary.

with time for various levels of pumping of the cross helicity. At $\chi = 0$, the time of attaining the quasi-steady state amounts to several vortex revolutions and, at $\chi = 0.6$, exceeds 100 dimensionless time units. In this case, both the average value of energy and the character of its oscillations vary.

The energy accumulation is also accompanied by cross-helicity accumulation. In Fig. 2, we show the average values of integral correlation coefficient $C = \langle H\rangle/\langle E^T\rangle$. It is substantial that the turbulence accumulates it at a low level of the introduced cross helicity ($\chi/\varepsilon \ll 1$); i.e., the integral correlation coefficient greatly exceeds the ratio between the introduced helicity and the introduced energy. Thus, this tendency is inherent not only to the Alfvén turbulence [7], but also to the isotropic (Kolmogorov) MHD turbulence. At large values of χ/ε, the coefficient C tends to unity.

Figure 3 shows how the energy spectra vary with increasing the level of the cross helicity introduced in the flow. We present the energy values for each scale compensated on the quantity $k_n^{2/3}$. In such a representation, it is the horizontal line that corresponds to the spectrum $k^{-5/3}$. It can be seen that there is such a spectral portion at $\chi = 0$, while both the energy of each scale and the spectrum slope increase with χ.

It is of interest to trace directly the time variation of the exchange. In Fig. 4, we show the vortex-revolution time and the energy and helicity-exchange times calculated for each envelope in the turbulence with a high level of cross helicity ($\chi/\varepsilon = 0.6$). It is indicative that the exchange time on the integral scale exceeds the vortex-revolution time by almost two orders of magnitude. This difference decreases with increasing the wave number and vanishes in the dissipative interval. In the inertial interval, the energy flux is also constant and the exchange time is unambiguously determined by the energy of pulsations of this scale; i.e., $t_n \sim \langle u_n^2\rangle$. This means that the power law for t_n coincides with the slope for the energy of pulsations. In the case shown in Fig. 4, $t_n \sim l_n^{0.89 \pm 0.02}$, and the energy distribution in the inertial interval follows the law $\langle u_n^2\rangle \sim l_n^{0.89 \pm 0.02}$ (the straight line in Fig. 3). The unexpected result is that the power law for the exchange time is retained also in the dissipative interval (Fig. 4). The helicity-exchange time behaves similarly to the energy-exchange time, but it is always somewhat less.

Thus, it is shown that the cross helicity hampers cascade energy transfer to MHD turbulence and results in energy accumulation in the system. This accumulation proceeds until the vortex intensification compensates for the decreasing efficiency of nonlinear interactions. The formula for estimating the average turbulence energy is obtained for the set ratio between the introduced helicity and the energy. It is of importance that the turbulence accumulates the introduced cross helicity at its low level—the integral correlation coefficient significantly exceeds the ratio between the introduced helicity and the energy. It is shown that the spectrum slope gradually increases from –5/3 to 2 with the cross-helicity level.

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